Critical-layer analysis of wind-driven oblique surface waves

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1   Introduction

The growth of wind-driven surface waves due to energy transfer from wind to wave was studied by Miles (1957). His analysis shows that the critical-layer dynamics play an important role in the growth of wind waves. The critical layer is located in air surrounding the critical height at which wind speed matches the wave speed. An initially small surface wave can grow exponentially as it draws energy from the wind.

Reutov (1980) studied the nonlinear effect in the wind-wave interaction by considering a plane wave that propagated along the downwind direction. However, the predicted maximum wave steepness was too small for the theory to be relevant for the ocean wave growth, as he pointed out.

Unlike the plane wave, the nonlinear interaction of oblique wind waves can enhance their growth rates as shown by Lee & Wundrow (2011) and Lee (2012). The nonlinear interaction between a pair of oblique wind waves of the same streamwise but opposite spanwise wavenumbers was investigated by Lee & Wundrow (2011). They show that the nonlinear interaction generates a large number of higher spanwise harmonics. All amplitudes grow very large and eventually become singular at a finite time.

Nonlinear interactions between two wind-driven oblique surface waves (larger primary wave and smaller secondary wave with different frequencies) of the same wave speed in the downwind direction were studied by Lee (2012). Numerical solutions in the inviscid and $O(1)$-viscosity cases show that the nonlinear growth rates become much larger than the linear growth rates.

In this study, the nonlinear interaction between two oblique wind waves will be investigated when the viscosity effect is larger than in Lee (2012). In the quasi-equilibrium critical layer of the present study, the mean-flow-convection effect is balanced with the viscous effect (instead of the wave-growth effect as in the non-equilibrium critical layer by Lee 2012).

2   Formulation

The mean flow in air is two-dimensional and there is no mean motion in water. The initial wave field is composed of a primary and secondary oblique wind waves (propagating obliquely to the

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downwind direction). The secondary-wave amplitude is smaller than the primary one. Initially they grow exponentially due to the energy transfer from wind by Miles' (1957) resonance mechanism. A right-handed Cartesian coordinate system (x, y, z) is attached to the calm-water surface with x in the streamwise direction of the wind, y in the spanwise lateral direction, and z in the vertical direction pointing up.

The air and water are assumed incompressible and the ratio of densities is defined as,

$$\rho_a/\rho_w = \sigma, \tag{1}$$

where the subscripts a and w denote the quantities of air and water, respectively, and $\sigma \ll 1$ characterizes the small density ratio of air to water.

The velocity potential of a primary wind wave, denoted by the letter P in Figure 1, is proportional to $\exp[\mathrm{i}\alpha_p(x - ct) + \mathrm{i}\beta_p y]$, where α_p and β_p are its streamwise and spanwise wavenumbers and c is the wave speed in the streamwise direction. That of a secondary wave, denoted by the letter S, is proportional to $\exp[\mathrm{i}\alpha_s(x - ct) + \mathrm{i}\beta_s y]$. The first nonlinear coupling between them occurs when the magnitude of the primary-wave steepness is $O(\sigma^3)$.

The nonlinear interaction between the primary and secondary wind waves generates a difference mode, which is proportional to $\exp[\mathrm{i}(\alpha_p - \alpha_s)(x - ct) + \mathrm{i}(\beta_p - \beta_s)y]$ and denoted by the letter D in Figure 1. The difference mode does not satisfy the free-surface-wave dispersion relation. The nonlinear coupling between the difference mode and primary wave induces a nonlinear growth of the secondary wind wave. During this initial nonlinear stage, the growth rate of the primary wind wave remains linear. This interaction involves two wind-driven surface waves and one difference mode.

3   Amplitude equation

The velocity potential $\tilde{\phi}$ in water can be written as,

$$\tilde{\phi} = A_p(t_1)\hat{\phi}_p(z, t_1)\mathrm{e}^{\mathrm{i}(\alpha_p X + \beta_p y)} + \delta A_s(t_1)\hat{\phi}_s(z, t_1)\mathrm{e}^{\mathrm{i}(\alpha_s X + \beta_s y)} + \delta\sigma A_d(t_1)\hat{\phi}_d(z, t_1)\mathrm{e}^{\mathrm{i}(\alpha_d X + \beta_d y)} + c.c., \tag{2}$$

where c.c. denotes the complex conjugates, $\mathrm{i} \equiv \sqrt{-1}$, and $\delta \ll 1$. The quantities of the primary, secondary and difference modes are denoted with the subscripts p, s and d, respectively. The time scale over which the wave growth occurs becomes $t_1 \equiv \sigma t$, and the normalized streamwise coordinate in a reference frame moving with the common wave speed c is given by $X \equiv x - ct$.

The amplitude equations can be obtained by matching the outer solutions with the nonlinear critical-layer solutions as,

$$A_p = a_p\, \mathrm{e}^{\bar{t}} \quad A_s = a_s\, \mathrm{e}^{\kappa_s \bar{t}}\, (1 - \mu_s\, \mathrm{e}^{2\bar{t}})^{\kappa_d}, \quad A_d = -a_s^*\mu_d\mathrm{e}^{(1+\kappa_s^*)\bar{t}}\, (1 - \mu_s^*\, \mathrm{e}^{2\bar{t}})^{\kappa_d^*}, \tag{3}$$

where $\bar{t}$ is a re-normalized variable of t_1, and a_p, a_s, κ_s, κ_d, μ_s and μ_d are constants.

Figure 2 shows the evolution of amplitudes when $\theta_p \equiv \arctan(\beta_p/\alpha_p) = 30^o$ and $\theta_s = -15^o$. The primary-wave amplitude A_p grows exponentially. In the linear stage where $\bar{t} < 3.5$, the secondary and difference modes, A_s and A_d, also grow exponentially (although the latter is nonlinearly generated). Once the primary mode amplitude reaches a certain level at $\bar{t} \approx 3.5$, it starts to affect the growth of the other modes. In the nonlinear-growth stage where $\bar{t} > 3.5$, both secondary and difference amplitudes grow much faster than the exponential ones (before they start to decay). The nonlinear growth rates near $\bar{t} \approx 4.5$ become very large.

It is worth noticing that the present study (with Lee & Wundrow 2011; Lee 2012) is the first to show that the nonlinear interaction between three-dimensional fluctuations in air, that are synchronized with oblique surface waves, is responsible for the faster than the exponential growth of wind waves.
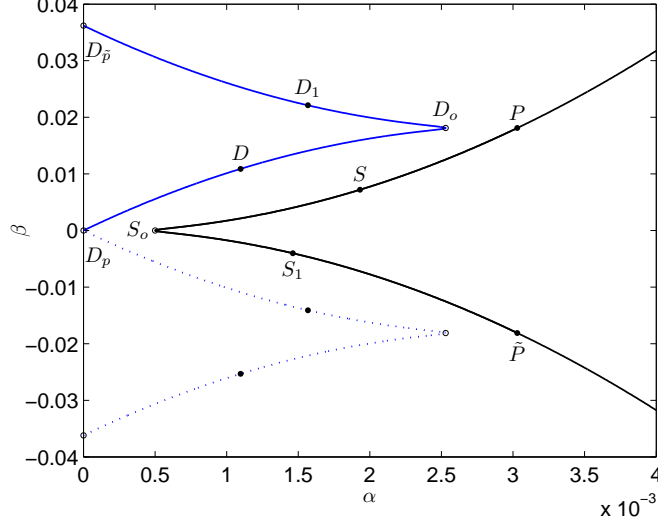
Figure 1: Three-mode interaction in (α, β) coordinates. Primary and secondary modes are located on black solid curve that depicts the free-surface-wave dispersion relation with constant $c = 10$.
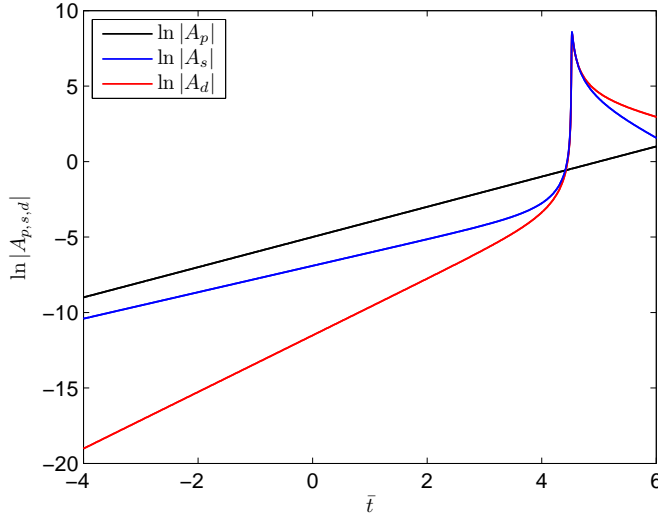

Figure 2: $\ln|A_p|$, $\ln|A_s|$ and $\ln|A_d|$ $vs.$ $\bar{t}$ when $\theta_p \equiv \tan^{-1}(\beta_p/\alpha_p) = 30^0$ and $\theta_s = -15^0$.

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